

THE CURVE SHORTENING FLOW WITH PARALLEL 1-FORM

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ABSTRACT. Let M be a closed Riemannian manifold with a parallel 1-form Ω . We prove two theorems about the curve shortening flow in M . One is that the curve shortening flow \mathcal{C}_t in M exists for all t in $[0, \infty)$, if it satisfies $\Omega(T) \geq 0$ on the initial curve \mathcal{C}_0 . Here T is the unit tangent vector on \mathcal{C}_0 . The other one is about the convergence. It says that in a closed Riemannian manifold \tilde{M} , assume the curve shortening flow \mathcal{C}_t exists for all $t \in [0, \infty)$ and its length converges to a positive limit, then $\lim_{t \rightarrow \infty} \max_{\mathcal{C}_t} |\nabla^m A|^2 = 0$ for all $m = 0, 1, \dots$. Here A denotes the second fundamental form of \mathcal{C}_t in \tilde{M} .

INTRODUCTION

We use ∇ denote the Levi-Civita connection of a Riemannian manifold. A 1-form Ω is called parallel if $\nabla\Omega = 0$. In this paper we consider the long time existence and the convergence of the curve shortening flow in the Riemannian manifold with arbitrary codimension. More precisely, the curve shortening flow \mathcal{C}_t is given as the solution of the follow equation:

$$(0.1) \quad \begin{cases} \frac{\partial \mathcal{C}_t}{\partial t} = \vec{H} \\ \mathcal{C}(x, 0) = \mathcal{C}_0(x); \end{cases}$$

Here $\vec{H} = \nabla_T T$ is the mean curvature vector of \mathcal{C}_t in the ambient Riemannian manifold.

The evolution of closed curves under (0.1) has received considerable study. For example, Gage ([**Gag84**]), Gage-Hamilton ([**GH86**]) and Grayson ([**Gra87**]), considered the evolution of the convex curve flowed by the mean curvature in the plane. Furthermore, Grayson ([**Gra89**]) generalized the results into the case on the embedded closed curve of Riemannian surfaces. Huisken ([**Hui98**]) and Andrews-Bryan ([**AB11**]) used the technique of distance comparison to investigate the curve shortening flow on the plane. For the mean curvature flow of the submanifolds with codimension $k \geq 2$, Andrews-Baker ([**AB10**]) proposed a machinery about the evolution of the second fundamental form A with arbitrary codimension. They also obtained the convergence of the mean curvature flow for the submanifold pinched to sphere in Euclidean space. More recently, with an integral curvature condition ([**KFLZ10**]) and a pinched condition in hyperbolic space form ([**KFLZ11**]) Liu, Xu, Ye and Zhao also proved two extension theorems for the mean curvature flow of the submanifold with high codimension. Smoczyk's survey ([**Smo12**]) is a good reference.

In a different direction, Wang ([**Wan02**]) initialized a way to use parallel n -forms Ω , $n \geq 2$, to investigate the graphic mean curvature flow for arbitrary codimension (≥ 2) in product manifolds. With the evolution equation of $*\Omega$, ($*$ is the Hodge dual), Wang obtained its long-time existence and an estimate of its second fundamental form along the mean curvature flow. Moreover, Wang and his coauthors generalized the results of graphic mean curvature flow into other settings ([**MW11**],[**TW04**],[**Wan01**],[**SW02**],). Both of ([**AB10**]) and ([**Wan02**]) did not consider the case of the curve shortening flow, which is the topic of this paper.

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There are two special properties about the curve shortening flow whose codimension in its ambient manifold is greater than 1. First the embeddedness of the evolving curve is not necessarily preserved ([Alt91]). Therefore, we only consider the case of the immersed curve. On the other hand, $|\tilde{H}|^2 = |A|^2$. Based on this fact, we can overcome the barrier that the curve shortening flow has arbitrary codimension. This makes many computations possible, especially in Section 2.

Now let us state two main theorems:

Theorem 0.1. *Let M be a closed Riemannian manifold, and Ω be a parallel 1-form in M . If C_0 is a closed curve satisfying $\Omega(T) > 0$ on C_0 . Here T is the unit tangent vector on C_0 . Then the curve shortening flow C_t in (0.1) exists for all t in $[0, \infty)$.*

Corollary 1. *Let (N, g) be a closed manifold, du^2 is the canonical metric on S^1 . Assume $M = S^1 \times N$ is the product manifold with the product metric $\bar{g} = g + du^2$. Let $\Omega = du$, which is a parallel 1-form in M . C_0 is a closed curve in M . If the unit tangent vector T of C_0 satisfies $\Omega(T) > 0$, the curve shortening flow C_t in (0.1) exists for all t in $[0, \infty)$.*

Remark 0.1. Corollary 1 can be viewed the 1-dimension version of Theorem A in ([Wan02]). The existence result here with $\Omega(T) > 0$ is stronger than that of graphical mean curvature flow with $*\Omega > \frac{1}{2}$ in ([Wan02]). It also can be compared with the results of Tsui-Wang ([TW04]) if we choose $N = S^n$.

Now we consider the convergence of the curve shortening flow in any closed Riemannian manifold. Notice that we do not require there is a parallel 1-form in the ambient manifold. The convergence of the curve shortening flow in the C^∞ sense can be stated as follows.

Theorem 0.2. *Let \tilde{M} be a closed manifold. If the curve shortening flow C_t in \tilde{M} exists for all $t \in [0, \infty)$ and the length of C_t converges to a positive number, $\lim_{t \rightarrow \infty} \max_{C_t} |\nabla^m A| = 0$ for all $m = 0, 1, \dots$.*

Remark 0.2. The definition of $\nabla^m A$ is very subtle. In ([Hui84], [Hui86]), Huisken's definitions about $\nabla^m A$ only work in the hypersurface's case. Andrews-Bakers ([AB10]) gave a rigorous definition of $\nabla^m A$ for any mean curvature flow with arbitrary codimension in Riemannian manifold. The basic idea is that $|\nabla^m A|$ can contain all information about the mean curvature flow. Namely, when $|\nabla^m A|$ goes to 0 when t goes to infinity, the mean curvature flow F_t will converge to a geodesic submanifold in some natural sense. For example, one of these cases is that F_t has long time existence and always stay in a compact region of its ambient Riemannian manifold. We will state the definition of $\nabla^m A$ in Section 2.

Remark 0.3. The condition that M and \tilde{M} are closed is not essential. We take them only for the sake of the exposition. In Theorem 0.1, what we really need is that for all finite time t , the curve shortening flow C_t is in a compact region in M , which can depend on time t . In Theorem 0.2, we can only require that all covariant derivatives of the Riemann tensor in \tilde{M} are uniformly bounded.

Outline of this paper. In section 1, we give two examples to illustrate our motivations of using parallel form to investigate the curve shortening flow. In Section 2, we give the definition of $\nabla^m A$. Then we derive the evolution equations related to the second fundamental form A (lemma 2.2 and lemma 2.3). With those equations, we derive the evolution equation of $\Omega(T)$ for any 1-form Ω along the curve shortening flow (lemma 2.4). In particular, when Ω is a parallel 1-form, this evolution equation states that the lower positive bound of $\Omega(T)$ can be preserved along the curve shortening flow. In section 3, we prove Theorem 0.1 by using the results from section 2. In section 4, with Grayson's idea ([Gra89]), we prove Theorem 0.2.

Notation. We collect some geometric quantities and facts used later.

- (1) $\mathcal{C}(u, t)$ denotes the solution of (0.1). $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial u}$ are the abbreviations of $\mathcal{C}_*(\frac{\partial}{\partial t})$ and $\mathcal{C}_*(\frac{\partial}{\partial u})$ respectively.
- (2) ∇ is the Levi-Civita connection, du^2 is the canonical metric on S^1 .
- (3) Let R denote Riemann curvature tensor, given by $R(X, Y) = \nabla_Y \nabla_X - \nabla_X \nabla_Y + \nabla_{[X, Y]}$.
- (4) Let s be the arc-length parameter of a curve \mathcal{C} . Its unit tangent vector T is equal to $\frac{\partial}{\partial s} = \frac{\mathcal{C}_*(\frac{\partial}{\partial u})}{\langle \mathcal{C}_*(\frac{\partial}{\partial u}), \mathcal{C}_*(\frac{\partial}{\partial u}) \rangle^{\frac{1}{2}}}$.
- (5) $\vec{H} = \frac{\partial}{\partial t} = A(T, T) = \nabla_T T$. $|A|^2 = |\vec{H}|^2$.
- (6) For a curve \mathcal{C} , $|\vec{H}|^2 = |A|^2 = \langle \nabla_T T, \nabla_T T \rangle$. All curves in this paper are closed.

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1. THE EXAMPLES ABOUT PARALLEL FORM

In this section we use two examples to illustrate the motivations that we choose parallel 1-form Ω to investigate the curve shortening flow. The first one is the graphic mean curvature flow in the product manifolds ([Wan02]). Based on this, we can use Wang's idea ([Wan02]) about parallel form to propose a problem about the open curve shortening flow in R^{1+p} .

Example 1 (graphic mean curvature flow). In ([Wan02]), Mu-Tao Wang initialized a way to use parallel form to deform a map along the mean curvature flow in product manifolds. The basic setting is as followed: Let N_1, N_2 be closed manifolds with constant sectional curvatures, dimension ≥ 2 . f is a smooth map from N_1 to N_2 . Let $F(p, t) : N_1 \times [0, q] \rightarrow N_1 \times N_2$ satisfies the following equation.

$$(1.1) \quad \begin{cases} \frac{\partial F_t}{\partial t} = \vec{H} \\ F(p, 0) = (p, f(p)); \end{cases}$$

Let Ω be the volume form of N_1 , which is parallel in the product manifold $N_1 \times N_2$. Together with some technical conditions about the sectional curvature of N_1 and N_2 , Wang ([Wan02]) obtained that if $*\Omega > \frac{1}{2}$ at the initial manifold, the evolution equations of $*\Omega$ indicates the long time existence of the solution in (1.1).

Example 2 (open curve shortening flow). In this example, we propose a question about open curve shortening flow in R^{1+p} . \mathcal{C}_0 in R^{1+p} is defined by

$$(1.2) \quad \mathcal{C}_0 : R^1 \rightarrow R^{1+p} \quad \text{and} \quad \mathcal{C}_0(x) = (x, f_1(x), \dots, f_p(x));$$

Let $|Df|^2 = \sum_{i=1, \dots, p} (f'_i(x))^2$. Then,

$$T = \left(\frac{\partial}{\partial x^1} + \sum_{i=1, \dots, p} f'_i(x) \frac{\partial}{\partial x^{i+1}} \right) \frac{1}{\sqrt{1 + |Df|^2}};$$

Let $\Omega = dx_1$. Again Ω is a parallel 1-form. On \mathcal{C}_0 , $\Omega(T) = \frac{1}{\sqrt{1 + |Df|^2}}$. From lemma 2.6, if \mathcal{C}_t exists for $t \in [0, q]$, then $\Omega(T)$ on \mathcal{C}_t satisfies

$$\frac{\partial}{\partial t} \Omega(T) = \Delta^{\mathcal{C}_t} \Omega(T) + |A|^2 \Omega(T)$$

Here A is its second fundamental form. Recall that with $*\Omega > \frac{1}{2}$ in the initial data, Wang ([Wan02]) answered the long time existence of graphic mean curvature flow positively. Naturally if we assume that \mathcal{C}_0 satisfies $\Omega(T) \geq \delta > 0$, does the curve shortening flow \mathcal{C}_t in (0.1) exist for all $t \in [0, \infty)$? If it exists, can we give any description about the convergence of this curve shortening flow in R^{1+p} ?

Remark 1.1. The above examples reflect our basic perspectives about the parallel form and the mean curvature flow. (2.6) and (3.3) are some possible evidences to support these connections between them. (2.6) can be viewed as a generalization of Wang's graphical mean curvature flow ([Wan02]) in the curve's case. (3.3) indicates that $\mu = \frac{1}{\Omega(T)}$ be thought as the gradient function ([EH89]) in the case of the curve shortening flow. There are no any oblivious connections between ([Wan02]) and ([EH89]). For the curve shortening flow, however, (3.3) and (2.6) are two different forms of the same evolution equation.

2. EVOLUTION EQUATIONS

We derive the evolution equations of the curve shortening flow. Generally, the forms of those equations are very complicated for the mean curvature flow whose dimension and codimension are both > 1 . ([AB10], [Wan02])

2.1. Definition of $\nabla^m A$. We briefly state the definition of $\nabla^m A$ which can be found in ([AB10]). Assume M is a Riemannian manifold with the Levi-Civita connection $\bar{\nabla}$, N is an immersed submanifold of M . Suppose I is a real interval, then the tangent space $T(N \times I)$ splits into a direct product $\mathcal{H} \otimes R\partial t$, where $\mathcal{H} = \{u \in T(N \times I); dt(u) = 0\}$ is the "spatial" tangent bundle.

We consider a smooth map $F : N \times I \rightarrow M$ which is a time-dependent immersion, i.e., for each fixed $t \in I$, $F(\cdot, t) : N \rightarrow M$ is an immersion. F^*TM is a vector bundle over $N \times I$. We can define a metric g_F and the connection ${}^F\nabla$ on F^*TM by the pull-back from $(\bar{\nabla}, \bar{g})$ on M . Let \mathcal{N} be the orthogonal complement of \mathcal{H} in F^*TM . We denote the π, π^\perp be the orthogonal projections from F^*TM onto \mathcal{H} and \mathcal{N} respectively. The connections ${}^{\mathcal{H}}\nabla$ and ${}^{\mathcal{N}}\nabla$ are given by.

$$\begin{aligned} {}^{\mathcal{H}}\nabla &= \pi \circ {}^F\nabla \circ F_*; \\ {}^{\mathcal{N}}\nabla &= \pi^\perp \circ {}^F\nabla \circ F_*; \end{aligned}$$

For a tensor $K \in \Gamma(\mathcal{H}^* \otimes \mathcal{H}^* \otimes N)$. We have to define $\nabla^m K \in \Gamma(\otimes^{m+2} \mathcal{H}^* \otimes N)$ by induction on m . Let $u_0, u_i, i = 1, 2, 3, \dots, m+1$, then $\nabla^m K$ and $\nabla_{\frac{\partial}{\partial t}} \nabla^m K$ are given by

$$\begin{aligned} \nabla_{u_0}(\nabla^{m-1} K)(u_1, \dots, u_{m+1}) &= {}^{\mathcal{N}}\nabla_{u_0}(\nabla^{m-1} K(u_1, \dots, u_{m+1})) - \sum_{i=1}^{m+1} \nabla^{m-1} K(\dots, {}^{\mathcal{H}}\nabla_{u_0} u_i, \dots); \\ \nabla_{\frac{\partial}{\partial t}}(\nabla^m K)(u_0, u_1, \dots, u_{m+1}) &= {}^{\mathcal{N}}\nabla_{\frac{\partial}{\partial t}}(\nabla^m K(u_0, \dots, u_{m+1})) - \sum_{k=0}^{m+1} \nabla^m K(\dots, {}^{\mathcal{H}}\nabla_{\frac{\partial}{\partial t}} u_k, \dots); \end{aligned}$$

2.2. The Evolution Equations for the Curve Shortening Flow. For the curve shortening flow \mathcal{C}_t in (0.1), we define a new tensor $\tilde{R} \in \Gamma(\mathcal{H}^* \otimes \mathcal{H}^* \otimes \mathcal{N})$ by $\tilde{R}(T, T) = R(T, \tilde{H})T$. We also have the definition of $\nabla^m \tilde{R}$. Since $A \in \Gamma(\mathcal{H}^* \otimes \mathcal{H}^* \otimes \mathcal{N})$, with the definitions in the previous subsection, we can obtain a theorem about the evolution equation of A from equation (18) in ([AB10]).

Theorem 2.1 (Andrews-Baker). *For the curve shortening flow \mathcal{C}_t , the evolution equation of $A(T, T)$ is given by*

$$(2.1) \quad \nabla_{\frac{\partial}{\partial t}} A(T, T) = \nabla_T \nabla_T A(T, T) + |A|^2 \tilde{H} + \tilde{R}(T, T);$$

Recall that $S * T$ ([Hui84]) means any linear combination of tensors formed by contraction on S and T . (2.1) can be rewritten as $\nabla_{\frac{\partial}{\partial t}} A = \nabla^2 A + \sum A * A * A + \tilde{R}(T, T)$. This form can be generalized into any m -th covariant derivative A for the curve shortening flow \mathcal{C}_t .

Lemma 2.2. *The evolution equation of m -th covariant derivative A is of the form.*

$$(2.2) \quad \nabla_{\frac{\partial}{\partial t}} \nabla^m A = \nabla^{m+2} A + \sum_{i+j+k=m} \nabla^i A * \nabla^j A * \nabla^k A + \nabla^m \tilde{R}(T, T);$$

Proof. We prove the lemma through the induction on m . The case $m = 0$ is given by (2.1). Now suppose the results hold up to $k \leq m - 1$. In ([Hui84]), the time derivative of the Christoffel symbols Γ_{jk}^i has the form $A * \nabla A$. Therefore we obtain,

$$\begin{aligned} \nabla_{\frac{\partial}{\partial t}} \nabla^m A &= \nabla(\nabla_{\frac{\partial}{\partial t}} \nabla^{m-1} A) + \nabla^{m-1} A * A * \nabla A; \\ &= \nabla\{\nabla^{m+1} A\} + \sum_{i+j+k=m-1} \nabla^i A * \nabla^j A * \nabla^k A + \nabla^{m-1} \tilde{R}(T, T) \\ &\quad + \nabla^{m-1} A * A * \nabla A; \\ &= \nabla^{m+2} A + \sum_{i+j+k=m} \nabla^i A * \nabla^j A * \nabla^k A + \nabla^m \tilde{R}(T, T); \end{aligned}$$

□

Moreover, we obtain the evolution equations of $|\nabla^m A|^2$, $m = 0, 1, \dots$.

Lemma 2.3. *The evolution equations of $|\nabla^m A|^2$ are given by*

$$(2.3) \quad \frac{\partial}{\partial t} |A|^2 = \triangle^{\mathcal{C}_t} |A|^2 - 2|\nabla A|^2 + 2|A|^4 + 2R(T, \vec{H}, T, \vec{H});$$

$$(2.4) \quad \begin{aligned} \frac{\partial}{\partial t} |\nabla^m A|^2 &= \triangle^{\mathcal{C}_t} |\nabla^m A|^2 - 2|\nabla^{m+1} A|^2 + \sum_{i+j+k=m} \nabla^i A * \nabla^j A * \nabla^k A * \nabla^m A \\ &\quad + 2 \langle \nabla^m \tilde{R}(T, T), \nabla^m A \rangle \quad \text{for } m \geq 1; \end{aligned}$$

Proof. We differentiate $|\nabla^m A|^2$ with respect to t . For $m = 0$,

$$\begin{aligned} \frac{\partial}{\partial t} |A|^2 &= 2 \langle \nabla_{\frac{\partial}{\partial t}} A(T, T), A(T, T) \rangle; \\ &= 2 \langle \nabla^2 A(T, T) + |A|^2 \vec{H} + R(T, \vec{H})T, A(T, T) \rangle; \end{aligned}$$

Since $A(T, T) = \vec{H}$

$$= \triangle^{\mathcal{C}_t} |A|^2 - 2|\nabla A|^2 + 2|A|^4 + 2R(T, \vec{H}, T, \vec{H});$$

For $m \geq 1$.

$$\begin{aligned} \frac{\partial}{\partial t} |\nabla^m A|^2 &= 2 \langle \nabla_{\frac{\partial}{\partial t}} \nabla^m A, \nabla^m A \rangle; \\ &= 2 \langle \nabla^{m+2} A + \sum_{i+j+k=m} \nabla^i A * \nabla^j A * \nabla^k A + \nabla^m \tilde{R}(T, T), \nabla^m A \rangle; \\ &= \triangle^{\mathcal{C}_t} |\nabla^m A|^2 - 2|\nabla^{m+1} A|^2 + \sum_{i+j+k=m} \nabla^i A * \nabla^j A * \nabla^k A * \nabla^m A + \\ &\quad + 2 \langle \nabla^m \tilde{R}(T, T), \nabla^m A \rangle; \end{aligned}$$

□

Now we derive the evolution equation of $\Omega(T)$ along the curve shortening flow \mathcal{C}_t .

Lemma 2.4. *Let Ω be a 1-form in a Riemannian manifold \tilde{M} , s be the arc-length parameter of \mathcal{C}_t . Let $\Delta^{C_t} = \frac{\partial^2}{\partial s^2}$ be the Laplace operator on \mathcal{C}_t , and $T = \frac{\partial}{\partial s}$ be the unit tangent vector. Thus we have*

$$(2.5) \quad \frac{\partial}{\partial t} \Omega(T) = \Delta^{C_t} \Omega(T) + |A|^2 \Omega(T) - \nabla^2 \Omega(T, T, T) - 2 \nabla \Omega(T, \nabla_T T);$$

In particular, if Ω is parallel in \tilde{M} , we obtain

$$(2.6) \quad \frac{\partial}{\partial t} \Omega(T) = \Delta^{C_t} \Omega(T) + |A|^2 \Omega(T);$$

Proof. Recall $T = \frac{C_*(\frac{\partial}{\partial u})}{\langle C_*(\frac{\partial}{\partial u}), C_*(\frac{\partial}{\partial u}) \rangle^{\frac{1}{2}}}$ and $\frac{\partial}{\partial t} = \vec{H}$. And

$$(2.7) \quad \begin{aligned} [\frac{\partial}{\partial t}, T] &= \frac{[C_*(\frac{\partial}{\partial t}), C_*(\frac{\partial}{\partial u})]}{\langle C_*(\frac{\partial}{\partial u}), C_*(\frac{\partial}{\partial u}) \rangle^{\frac{1}{2}}} + C_*(\frac{\partial}{\partial u}) \frac{\partial}{\partial t} \left(\frac{1}{\langle C_*(\frac{\partial}{\partial u}), C_*(\frac{\partial}{\partial u}) \rangle^{\frac{1}{2}}} \right); \\ &= -C_*(\frac{\partial}{\partial u}) \frac{\langle \nabla_{C_*(\frac{\partial}{\partial t})}(C_*(\frac{\partial}{\partial u})), C_*(\frac{\partial}{\partial u}) \rangle}{\langle C_*(\frac{\partial}{\partial u}), C_*(\frac{\partial}{\partial u}) \rangle^{\frac{3}{2}}}; \\ &= -C_*(\frac{\partial}{\partial u}) \frac{\langle \nabla_{C_*(\frac{\partial}{\partial u})}(\vec{H}), C_*(\frac{\partial}{\partial u}) \rangle}{\langle C_*(\frac{\partial}{\partial u}), C_*(\frac{\partial}{\partial u}) \rangle^{\frac{3}{2}}}; \\ &= |A|^2 T; \end{aligned}$$

Therefore, the relation of $\nabla_{\frac{\partial}{\partial t}} T$ and $\nabla_T \frac{\partial}{\partial t}$ is given by

$$(2.8) \quad \nabla_{\frac{\partial}{\partial t}} T - \nabla_T \frac{\partial}{\partial t} = |A|^2 T;$$

$$(2.9) \quad \begin{aligned} \Delta^{C_t} \Omega(T) &= T(\Omega(\nabla_T T) + \nabla \Omega(T, T)); \\ &= \Omega(\nabla_T \frac{\partial}{\partial t}) + \nabla \Omega(\frac{\partial}{\partial t}, T) + \nabla^2 \Omega(T, T, T) + 2 \nabla \Omega(T, \nabla_T T); \end{aligned}$$

The t -derivative of $\Omega(T)$

$$(2.10) \quad \begin{aligned} \frac{d}{dt} \Omega(T) &= \Omega(\nabla_{\frac{\partial}{\partial t}} T) + \nabla \Omega(\frac{\partial}{\partial t}, T); \\ &= \Omega(\nabla_T \frac{\partial}{\partial t}) + |A|^2 \Omega(T) + \nabla \Omega(\frac{\partial}{\partial t}, T); \end{aligned}$$

Therefore, (2.9) and (2.10) lead to the lemma. \square

Remark 2.1. It's easily checked $\min_{\mathcal{C}_t} \Omega$ is a Lipschitz function of t , and $\min_{\mathcal{C}_t} \Omega(T)$ is differentiable with respect to t almost everywhere. When Ω is a parallel 1-form in \tilde{M} , $\Omega(T) > 0$ for $t \in [0, q_0)$. For $\Delta^{C_t} \Omega(T) \geq 0$ for the points which attain the minimal value of $\Omega(T)$, for a.e $t \in [0, q_0)$ we have the following:

$$\frac{d}{dt} \min_{\mathcal{C}_t} \Omega(T) \geq |A|^2 \min_{\mathcal{C}_t} \Omega(T) \geq 0;$$

This implies that $\min_{\mathcal{C}_t} \Omega(T)$ is nondecreasing along the flow. Then $\Omega(T) \geq \delta_0$ will be preserved whenever the curve shortening flow exists.

3. LONG TIME EXISTENCE

With the evolution results in the previous section, we prove Theorem 0.1. The short time existence of \mathcal{C}_t is from the short time existence of solution of the quasilinear parabolic equation for closed initial data. It's well known that if the mean curvature flow F_t of the hypersurface in Euclidean space exists for only finite time interval $[0, q)$, $\max_{F_t}|A| \rightarrow \infty$ when t approach to q ([Hui84]). For the submanifold in Euclidean space with high codimension, such kind of lemma is proved by Andrews-Baker ([AB10]). For the curve shortening flow's case, we give the proof of the following lemma only for the sake of completeness.

Lemma 3.1. *Let \tilde{M} be a Riemannian manifold. If the curve shortening flow \mathcal{C}_t in \tilde{M} exists for t in the maximal finite interval $[0, q)$, then $\max_{\mathcal{C}_t}|A|^2 \rightarrow \infty$ as $t \rightarrow q$.*

Proof. If the lemma is false, there exists a constant $C_4 < \infty$ such that $\max_{\mathcal{C}_t}|A| \leq C_4$ for $t \in [0, q)$. It follows that for all $u \in S^1$ and $t_1, t_2 \in [0, q)$.

$$\begin{aligned} \text{dist}(\mathcal{C}_{t_1}(u), \mathcal{C}_{t_2}(u)) &\leq \left| \int_{t_1}^{t_2} \vec{H} dt \right| \leq C_4 |t_1 - t_2|; \\ \frac{d}{dt} \int_{\mathcal{C}_t} ds_t &\geq -C_4 \int_{\mathcal{C}_t} ds_t; \end{aligned}$$

Therefore, \mathcal{C}_t converges uniformly to some continuous limit $\mathcal{C}_q(u)$ and the length $l_{\mathcal{C}_t} \geq \delta > 0$ for $t \in [0, q)$. We want to show that $\mathcal{C}_q(u)$ actually represents a smooth limit curve. Then we can extend the flow \mathcal{C}_t over time q . It's a contradiction to the maximal finite interval in the assumption.

By (2.4), we are led to

$$\begin{aligned} (3.1) \quad \frac{\partial}{\partial t} |\nabla^m A|^2 &\leq \Delta^{\mathcal{C}_t} |\nabla^m A|^2 - 2|\nabla^{m+1} A|^2 + C_m \sum_{i+j+k=m} |\nabla^i A| |\nabla^j A| |\nabla^k A| |\nabla^m A| \\ &\quad + D_m \sum_{j \leq m} |\nabla^j A| |\nabla^m A| + \tilde{C}_m |\nabla^m A|; \end{aligned}$$

By Cauchy inequality, we have the follows.

$$\begin{aligned} (3.2) \quad |\nabla^m A|^2 &\leq \min_{\mathcal{C}_t} |\nabla^m A|^2 ds_t + \int_{\mathcal{C}_t} \frac{\partial}{\partial s} |\nabla^m A|^2 ds_t; \\ &\leq \frac{1}{l_{\mathcal{C}_t}} \int |\nabla^m A|^2 ds_t + 2 \int_{\mathcal{C}_t} \langle \nabla^m A, \nabla^{m+1} A \rangle ds_t; \\ &\leq \left(\frac{1}{\delta} + 2\right) \int |\nabla^m A|^2 ds_t + 2 \int_{\mathcal{C}_t} |\nabla^{m+1} A|^2 ds_t; \end{aligned}$$

If \mathcal{C}_t satisfies $|\nabla^m A| \leq C_m$ for all $t < q$, then \mathcal{C}_q is a smooth curve. From our definition about $\nabla^m A$, the proof of this fact is classical but a little tedious. Therefore we omit it here. For the Euclidean case, please refer to the proof of Theorem 3 in ([AB10]).

By (3.2), we have to prove that $\int_{\mathcal{C}_t} |\nabla^m A|^2 ds_t \leq C_m$ for all m . We argue $|\int_{\mathcal{C}_t} \nabla^m A ds_t| \leq C_m$ by induction on m . If it holds up for $k \leq m-1$. Then by (3.2) for $k \leq m-2$, $\max_{\mathcal{C}_t} |\nabla^k A| \leq C_k$ for $t \in [0, q)$. (3.1) and (3.2) imply that

$$\frac{\partial}{\partial t} \int_{\mathcal{C}_t} |\nabla^m A|^2 ds_t \leq \tilde{C}(m) \left(\int_{\mathcal{C}_t} |\nabla^m A|^2 ds_t + \left(\int_{\mathcal{C}_t} |\nabla^m A|^2 ds_t \right)^{\frac{1}{2}} \right);$$

With the above inequality, we obtain

$$\left(\int_{\mathcal{C}_t} |\nabla^m A|^2 ds_t \right)^{\frac{1}{2}} + 1 \leq e^{\frac{\tilde{C}(m)}{2}t} \left(\left(\int_{\mathcal{C}_0} |\nabla^m A|^2 ds_0 \right)^{\frac{1}{2}} + 1 \right);$$

Then we conclude that for all $m = 1, \dots, \max_{\mathcal{C}_t} |\nabla^m A| \leq C_m$. $C_q(u)$ is the smooth limit curve of \mathcal{C}_t for $t < q$. This will lead to the contradiction, since we suppose $[0, q)$ is the maximal time interval for the existence of \mathcal{C}_t . \square

Now let $\mu = \frac{1}{\Omega(T)}$. The following lemma tells us μ can be viewed as the gradient function ν in ([EH89]). They have the same form of evolution equation.

Lemma 3.2. *Let M be a closed manifold, Ω be a parallel 1-form in M , s be the arc-length parameter of \mathcal{C}_t . If the curve shortening flow \mathcal{C}_t exists for $t \in [0, q)$ and \mathcal{C}_0 satisfies $\Omega(T) > 0$, then μ satisfies*

$$(3.3) \quad \left(\frac{d}{dt} - \Delta^{\mathcal{C}_t}\right)\mu = -|A|^2\mu - 2\mu^{-1}\left|\frac{\partial\mu}{\partial s}\right|^2;$$

Proof.

$$\begin{aligned} \left(\frac{d}{dt} - \Delta^{\mathcal{C}_t}\right)\mu &= -\frac{1}{\Omega(T)^2}\left(\frac{d}{dt} - \Delta^{\mathcal{C}_t}\right)\Omega(T) - 2\Omega(T)\left|\frac{\partial\Omega(T)}{\partial s}\right|; \\ &= -\frac{1}{\Omega(T)^2}(|A|^2\Omega(T)) - 2\Omega(T)\left|\frac{\partial}{\partial s}\frac{1}{\Omega(T)}\right|^2; \\ &= -|A|^2\mu - 2\mu^{-1}\left|\frac{\partial\mu}{\partial s}\right|^2; \end{aligned}$$

\square

Lemma 3.3. *Use the assumption in lemma 3.2. Let $C_0 = \max_{x \in M} R$. If the curve shortening flow \mathcal{C}_t exists for $t \in [0, q)$, then*

$$(3.4) \quad \left(\frac{d}{dt} - \Delta^{\mathcal{C}_t}\right)|A|^2\mu^2 \leq -2\mu^{-1}\frac{\partial\mu}{\partial s}\frac{\partial|A|^2\mu^2}{\partial s} + 2C_0|A|^2\mu^2;$$

Proof.

$$\left(\frac{d}{dt} - \Delta^{\mathcal{C}_t}\right)|A|^2 = -2|\nabla A|^2 + 2|A|^4 + 2R(T, \vec{H}, T, \vec{H});$$

and together with the identity

$$\left(\frac{d}{dt} - \Delta^{\mathcal{C}_t}\right)\mu^2 = -2|A|^2\mu^2 - 6\left|\frac{\partial\mu}{\partial s}\right|^2$$

yields

$$(3.5) \quad \left(\frac{d}{dt} - \Delta^{\mathcal{C}_t}\right)|A|^2\mu^2 \leq -2|\nabla A|^2\mu^2 - 6\left|\frac{\partial\mu}{\partial s}\right|^2|A|^2 - 2\frac{\partial|A|^2}{\partial s}\frac{\partial\mu^2}{\partial s} + 2R(T, \vec{H}, T, \vec{H})\mu^2;$$

and

$$\begin{aligned} -2\frac{\partial|A|^2}{\partial s}\frac{\partial\mu^2}{\partial s} &\leq -\frac{\partial|A|^2}{\partial s}\frac{\partial\mu^2}{\partial s} - 4\mu|A|\frac{\partial|A|}{\partial s}\frac{\partial\mu}{\partial s}; \\ (3.6) \quad &\leq -\mu^{-2}\frac{\partial\mu^2}{\partial s}\frac{\partial|A|^2\mu^2}{\partial s} + 4\left|\frac{\partial\mu}{\partial s}\right|^2|A|^2 - 4\mu|A|\frac{\partial|A|}{\partial s}\frac{\partial\mu}{\partial s} \\ &\leq -2\mu^{-1}\frac{\partial\mu}{\partial s}\frac{\partial|A|^2\mu^2}{\partial s} + 2\left|\frac{\partial|A|}{\partial s}\right|^2\mu^2 + 6\left|\frac{\partial\mu}{\partial s}\right|^2|A|^2; \end{aligned}$$

Here we use the fact $\left|\frac{\partial|A|}{\partial s}\right| \leq |\nabla A|$. For M is closed, $|\vec{H}|^2 = |A|^2$, $|R(T, \vec{H}, T, \vec{H})| \leq C_0|A|^2$. By (3.5) and (3.6), we obtain

$$(3.7) \quad \left(\frac{d}{dt} - \Delta^{\mathcal{C}_t}\right)|A|^2\mu^2 \leq -2\mu^{-1}\frac{\partial\mu}{\partial s}\frac{\partial|A|^2\mu^2}{\partial s} + 2C_0|A|^2\mu^2;$$

\square

Corollary 2. *Under the assumption of Theorem 0.1. Then $\max_{\mathcal{C}_t} |A| \leq C(q, C_0)$ when $t < q$. Here $C(q, C_0)$ is a constant only dependent on q and C_0 .*

Proof. Because M is closed, for any unit vector field S in TM , $|\Omega(S)| \leq C_M$.

$$\mu^{-1} \frac{\partial \mu}{\partial s} = \Omega(T) \frac{\partial}{\partial s} \frac{1}{\Omega(T)} = -\mu \Omega(\vec{H});$$

Since $|\mu \Omega(\vec{H})|$ is continuous for $(u, t) \in S^1 \times [0, q)$, the maximal principle of parabolic equation in (3.7) gives ,

$$(3.8) \quad \frac{\partial}{\partial t} \max_{\mathcal{C}_t} |A|^2 \mu^2 \leq 2C_0 \max_{\mathcal{C}_t} |A|^2 \mu^2;$$

From $\mu \geq \frac{1}{C_M}$ and (3.8), we conclude the corollary. \square

The proof of Theorem 1. Now we prepare everything which we need to prove Theorem 0.1. From corollary 2, for all finite time interval $[0, q)$, $\max_{\mathcal{C}_t} |A|$ is always uniformly bounded (depending on q). If \mathcal{C}_t exists only for a finite interval, Lemma 3.1 will lead to a contraction. Then Theorem 1 is followed.

4. THE CONVERGENCE

This section is devoted to prove Theorem 0.2. Recall Theorem 0.2 concludes that the limit of $\max_{\mathcal{C}_t} |\nabla^m A|$ is 0 for all $m = 0, 1, \dots$ if the curve shortening flow \mathcal{C}_t in the closed manifold \tilde{M} exists for all $t \in [0, \infty)$ and the length of \mathcal{C}_t converges to a positive number. The idea of its proof originated from Section 7 in ([Gra89]). We suppose the length of \mathcal{C}_t satisfies $l_{\mathcal{C}_t} \geq l_\infty > 0$. First, let's prove a technique lemma.

Lemma 4.1. *For all $m \geq 0$,*

$$(4.1) \quad |\nabla^m A|^2 \leq a \int |\nabla^m A|^2 ds_t + 2 \int |\nabla^{m+1} A|^2 ds_t;$$

Here $a = \frac{1}{l_\infty} + 2$. For $m \geq 1$,

$$(4.2) \quad \left(\int |\nabla^m A|^2 ds_t \right)^2 \leq \int |\nabla^{m-1} A|^2 ds_t \int |\nabla^{m+1} A|^2 ds_t;$$

Proof. For $|\nabla^m A|^2$,

$$\begin{aligned} |\nabla^m A|^2 &\leq \min_{\mathcal{C}_t} |\nabla^m A|^2 ds_t + \int_{\mathcal{C}_t} \frac{\partial}{\partial s} |\nabla^m A|^2 ds_t; \\ &\leq \frac{1}{l_{\mathcal{C}_t}} \int |\nabla^m A|^2 ds_t + 2 \int_{\mathcal{C}_t} \langle \nabla^m A, \nabla^{m+1} A \rangle ds_t; \\ &\leq \left(\frac{1}{l_{\mathcal{C}_t}} + 2 \right) \int |\nabla^m A|^2 ds_t + 2 \int_{\mathcal{C}_t} |\nabla^{m+1} A|^2 ds_t; \end{aligned}$$

Integrating $\int |\nabla^m A|^2 ds_t$ by parts,

$$\int |\nabla^m A|^2 ds_t = \int \frac{\partial}{\partial s} \langle \nabla^{m-1} A, \nabla^m A \rangle ds_t - \int \langle \nabla^{m-1} A, \nabla^{m+1} A \rangle ds_t;$$

The first term is 0. Using the Cauchy inequality we obtain (4.2). \square

The proof of Theorem 0.2. Our strategy is to prove for all $m = 0, 1, \dots$, $\int |\nabla^m A|^2 ds_t$ converges to 0 as $t \rightarrow \infty$. First, we prove $m = 0$ case, then argue by induction.

From the definition of the curve shortening flow, we get the following two facts.

$$(4.3) \quad \frac{d}{dt} \int_{\mathcal{C}_t} ds_t = - \int_{\mathcal{C}_t} |\vec{H}|^2 ds_t = - \int_{\mathcal{C}_t} |A|^2 ds_t;$$

$$(4.4) \quad l_{\mathcal{C}_t} - l_{\mathcal{C}_0} = \int_0^t \frac{d}{dt} \int_{\mathcal{C}_t} ds_t dt = - \int_0^t \int_{\mathcal{C}_t} |A|^2 ds_t dt < \infty;$$

There exists a measure zero set E in $[0, \infty)$. Let $E_n = [n, \infty) \cap E^c$. We get

$$(4.5) \quad \max_{t \in E_n} \int_{\mathcal{C}_t} |A|^2 ds_t \rightarrow 0 \quad \text{as } n \rightarrow \infty;$$

From (2.3), we differentiate $\int_{\mathcal{C}_t} |A|^2 ds_t$ with respect to t .

$$(4.6) \quad \begin{aligned} \frac{d}{dt} \int_{\mathcal{C}_t} |A|^2 ds_t &\leq -2 \int_{\mathcal{C}_t} |\nabla A|^2 ds_t + 2 \int_{\mathcal{C}_t} |A|^4 ds_t + D_0 \int_{\mathcal{C}_t} |A|^2 ds_t; \\ &\leq -2 \int_{\mathcal{C}_t} |\nabla A|^2 ds_t + (a \int_{\mathcal{C}_t} |A|^2 ds_t + 2 \int_{\mathcal{C}_t} |\nabla A|^2 ds_t) \int_{\mathcal{C}_t} |A|^2 ds_t \\ &\quad + D_0 \int_{\mathcal{C}_t} |A|^2 ds_t; \\ &\leq -(2 - 2 \int_{\mathcal{C}_t} |A|^2 ds_t) \int_{\mathcal{C}_t} |\nabla A|^2 ds_t + \int_{\mathcal{C}_t} |A|^2 ds_t (D_0 + a \int_{\mathcal{C}_t} |A|^2 ds_t); \end{aligned}$$

Here we use (4.1). Let n be a sufficiently large number. We can assume $(2 - 2 \int_{\mathcal{C}_t} |A|^2 ds_t) \geq 1$ and $D_0 + a \int_{\mathcal{C}_t} |A|^2 ds_t \leq D_0 + 1$ for all $t \in E_n$. Furthermore,

$$(4.7) \quad \frac{d}{dt} \int_{\mathcal{C}_t} |A|^2 ds_t \leq (D_0 + 1) \int_{\mathcal{C}_t} |A|^2 ds_t$$

For both sides of (4.7) are continuous with respect to t , (4.7) holds up for $t \in [n, \infty)$. $\int_{\mathcal{C}_t} |A|^2 ds_t$ increases at most with exponential growth when n is sufficiently large. By (4.5), we obtain $\lim_{t \rightarrow \infty} \int_{\mathcal{C}_t} |A|^2 ds_t = 0$.

We notice that \tilde{M} is closed, all $|\nabla^m R| \leq C_m$ for all m . Together with (2.4), we have the following inequality for $m = 1, \dots$.

$$(4.8) \quad \begin{aligned} \frac{\partial}{\partial t} |\nabla^m A|^2 &\leq \Delta^{\mathcal{C}_t} |\nabla^m A|^2 - 2 |\nabla^{m+1} A|^2 \\ &\quad + \tilde{C}_m \sum_{i+j+k=m} |\nabla^i A| |\nabla^j A| |\nabla^k A| |\nabla^m A| + D_m \sum_{j \leq m} |\nabla^j A| |\nabla^m A| + \tilde{C}_m |\nabla^m A|; \end{aligned}$$

Here all constants in this proof are only depending on C_m .

When $m \geq 1$, we argue by the induction on m . Suppose $\lim_{t \rightarrow \infty} \int_{\mathcal{C}_t} |\nabla^k A|^2 ds_t = 0$ for $k \leq m-1$. Then by (4.1), for $k \leq m-2$, $\lim_{t \rightarrow \infty} \max_{\mathcal{C}_t} |\nabla^k A|^2 = 0$.

For any $\epsilon > 0$, let t_0 be a positive number sufficiently large, such that for $t \geq t_0$ and $k \leq m-2$, $\int_{\mathcal{C}_t} |\nabla^{m-1} A|^2 ds_t \leq \epsilon$ and $\max_{\mathcal{C}_t} |\nabla^k A|^2 \leq \epsilon$. By (4.8), we obtain the following estimates.

$$\begin{aligned} \frac{\partial}{\partial t} |\nabla^m A|^2 &\leq \triangle_{\mathcal{C}_t} |\nabla^m A|^2 - 2|\nabla^{m+1} A|^2 + \tilde{C}_m(\epsilon)(|\nabla^m A|^2 \\ &\quad + |\nabla^m A| |\nabla^{m-1} A|) + \tilde{D}_m |\nabla^m A|; \\ \frac{d}{dt} \int_{\mathcal{C}_t} |\nabla^m A|^2 ds_t &\leq -2 \int_{\mathcal{C}_t} |\nabla^{m+1} A|^2 ds_t + \tilde{C}_m(\epsilon) \left(\int_{\mathcal{C}_t} |\nabla^m A|^2 ds_t \right. \\ &\quad \left. + \int_{\mathcal{C}_t} |\nabla^m A| |\nabla^{m-1} A| ds_t \right) + \tilde{D}_m \int_{\mathcal{C}_t} |\nabla^m A| ds_t; \end{aligned}$$

Here $\tilde{C}_m(\epsilon)$ goes to 0 as ϵ converges to 0.

Let $C > 3$ be a constant determined later. If $\int_{\mathcal{C}_t} |\nabla^m A|^2 ds_t \geq C \int_{\mathcal{C}_t} |\nabla^{m-1} A|^2 ds_t$, the following estimates are given by (4.2).

$$\begin{aligned} \int_{\mathcal{C}_t} |\nabla^m A|^2 ds_t &\leq \frac{1}{C} \int_{\mathcal{C}_t} |\nabla^{m+1} A|^2 ds_t; \\ \int_{\mathcal{C}_t} |\nabla^{m-1} A|^2 ds_t &\leq \frac{1}{C^2} \int_{\mathcal{C}_t} |\nabla^{m+1} A|^2 ds_t; \end{aligned} \quad (4.9)$$

As a result,

$$\begin{aligned} \int_{\mathcal{C}_t} |\nabla^m A| |\nabla^{m-1} A| ds_t &\leq \frac{1}{\sqrt{C^3}} \int_{\mathcal{C}_t} |\nabla^{m+1} A|^2 ds_t; \\ \int_{\mathcal{C}_t} |\nabla^m A| ds_t &\leq \frac{\sqrt{l_{\mathcal{C}_0}}}{\sqrt{C}} \left(\int_{\mathcal{C}_t} |\nabla^{m+1} A|^2 ds_t \right)^{\frac{1}{2}}; \end{aligned} \quad (4.10)$$

The t -derivative of $\int_{\mathcal{C}_t} |\nabla^m A|^2 ds_t$ can be written as followed:

$$\begin{aligned} \frac{d}{dt} \int_{\mathcal{C}_t} |\nabla^m A|^2 ds_t &\leq -(2 - \tilde{C}_m(\epsilon) \left(\frac{1}{\sqrt{C^3}} + \frac{1}{C} \right)) \int_{\mathcal{C}_t} |\nabla^{m+1} A|^2 ds_t \\ &\quad + \frac{\tilde{D}_m}{\sqrt{C}} \left(\int_{\mathcal{C}_t} |\nabla^{m+1} A|^2 ds_t \right)^{\frac{1}{2}}; \end{aligned} \quad (4.11)$$

Let C be sufficiently large such that $(2 - \tilde{C}_m(\epsilon) \left(\frac{1}{\sqrt{C^3}} + \frac{1}{C} \right)) \geq 1$ and $\frac{\tilde{D}_m}{\sqrt{C}} \leq 2\epsilon$. (4.11) becomes

$$\frac{d}{dt} \int_{\mathcal{C}_t} |\nabla^m A|^2 ds_t \leq - \int_{\mathcal{C}_t} |\nabla^{m+1} A|^2 ds_t)^{\frac{1}{2}} \left(\int_{\mathcal{C}_t} |\nabla^{m+1} A|^2 ds_t \right)^{\frac{1}{2}} - 2\epsilon; \quad (4.12)$$

Now we can conclude Theorem 0.2 as follows.

- (1) If $\int_{\mathcal{C}_t} |\nabla^m A|^2 ds_t \geq C \int_{\mathcal{C}_t} |\nabla^{m-1} A|^2 ds_t$,
- (a) If $(\int_{\mathcal{C}_t} |\nabla^m A|^2 ds_t)^{\frac{1}{2}} \geq 3\epsilon$, we have $(\int_{\mathcal{C}_t} |\nabla^{m+1} A|^2 ds_t)^{\frac{1}{2}} \geq 3\epsilon$. From (4.12), then $\frac{d}{dt} \int_{\mathcal{C}_t} |\nabla^m A|^2 ds_t \leq -2\epsilon^2$. $\int_{\mathcal{C}_t} |\nabla^m A|^2 ds_t$ will decrease until

$$\left(\int_{\mathcal{C}_t} |\nabla^m A|^2 ds_t \right) \leq (3\epsilon)^2$$

If we can find a t_0 such that

$$\left(\int_{\mathcal{C}_t} |\nabla^m A|^2 ds_t \right)^{\frac{1}{2}} \leq 3\epsilon;$$

then for all $t > t_0$, $\int_{\mathcal{C}_t} |\nabla^m A|^2 ds_t \leq (3\epsilon)^2$.

- (b) If $(\int_{\mathcal{C}_t} |\nabla^m A|^2 ds_t)^{\frac{1}{2}} \leq 3\epsilon$, the conclusion of theorem 0.2 is obviously true.

(2) If $\int_{C_t} |\nabla^m A|^2 ds_t < C \int_{C_t} |\nabla^{m-1} A|^2 ds_t$, we have $\int_{C_t} |\nabla^m A|^2 ds_t \leq C\epsilon$.

In a word, finally $\int_{C_t} |\nabla^m A|^2 ds_t$ converges to 0 when $t \rightarrow \infty$. By (4.1), $\max_{C_t} |\nabla^m A|$ converges to 0 for all m as t goes to ∞ .

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